Int. J. Solids Structures Vol. 29, No. 7, pp. 845–853, 1992
Printed in Great Britain.

0020-7683/92 $5.00 + .00
Pergamon Press plc

TORSIONAL STRESS IN TUBULAR LAP JOINTS

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(*Received* 3 *December* 1990; *in revised form* 30 *June* 1991)

Abstract—In the current paper, the stress distribution in adhesive-bonded tubular lap joints subjected to torsion is analyzed. The two adherends may have different thicknesses and consist of different materials and the adhesive layer may be flexible or inflexible. The analysis is based on the elasticity theory in conjunction with the variational principle of complementary energy. By means of the present approach, closed form solutions are obtained and the stress-free end conditions of the joint which may have significant effect on the stress intensities is satisfied. Special attention is given to the high stress intensities in the end zones of the joint, and a stress concentration factor is deduced.

INTRODUCTION

With the development of high-strength adhesive materials and with the progress in techniques of adhesive bonding, various kinds of adhesive-bonded joints are now being used in the manufacturing of light structures (Kinloch, 1987). Because stress concentration often occurs in the edge zones of the adhesive layer of a joint (Goland and Reissner, 1944; Chen and Cheng, 1983, 1990, 1991), a detailed analysis of the stress distribution around the joint region, especially in the adhesive layer of these joints, is needed for application and research.

The problem of torsional stresses in tubular lap joints was first investigated by Volkersen (1965a). In Volkersen's analysis (1965b), the two tubular adherends of the joint are treated by Mechanics of Materials approach for which the circumferential shear stress $\tau_{r\theta}$ is disregarded and the adhesive layer is treated as a kind of "shearing spring" acting between the two adherends. Following Volkersen's work (1965c), Adams and Peppiatt (1977a) improved the analysis by taking the thickness of the adhesive layer into account, and Chon (1982) extended Volkersen's approach to tubular lap joints with adherends of composite materials. Attention, however, should be paid to the limitation of Volkersen's method (1965d), for in such an approach, where the existence of the circumferential shear stress $\tau_{r\theta}$ (cf. Fig. 1) in the two tubular adherends is disregarded, we may reasonably expect that some significant deviations in stress intensity could result for those joint combinations in which the adhesive layer is relatively thin and "stiff" (compared with the adherends).

The current work presents a more accurate approach by taking the circumferential shear stresses in the two tubular adherends into consideration. Through the use of the variational principle of complementary energy, a general formula for the stress distribution in the adhesive layer, which is suitable for arbitrary adherend–adhesive combinations is derived. The formula suggests that Volkersen–Adams' (1965e, 1977b) prediction for the stress concentration in the adhesive layer of a tubular lap joint under torsion is in fact more serious than it actually is, especially for those joint combinations where the adhesive layer is thin and "stiff" when compared with the adherends.

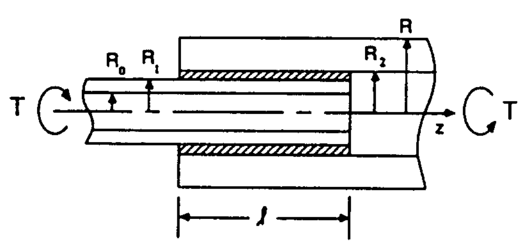

Fig. 1. Adhesive-bonded tubular lap.

FORMULATION OF THE PROBLEM

Figure 1 is a schematic diagram of a tubular lap joint subjected to torsion. The two tubular adherends of the joint may have unequal shear moduli G_1 (inner adherend) and G_2 (outer adherend) and the shear modulus for the adhesive layer is G_3. Let the inner and outer radii for the inner adherend be R_0 and R_1, and for the outer adherend, R_2 and R. The thickness of the adhesive layer is then $t = R_2 - R_1$. In Fig. 1, the two applied torques, equal and opposite, acting on the far ends of the two adherends respectively, are represented by T, and l is the length of the lap region. As mentioned above, the main concern is the stress distribution in the adhesive layer, especially the stress intensity in the end zones of the adhesive layer.

To solve the problem the jointed portion of the tubular lap joint is separated from the joint as shown in Fig. 2. To fit the geometry of the joint, a circular cylindrical coordinate system (r, θ, z) is chosen for describing the stresses in the joint block. The left end face of the block is identified by $z = 0$ (Fig. 2).

DESCRIPTION OF STRESSES IN THE JOINT

For the present problem, in view of the symmetric property in geometry and loading, we may assume that there are only two stress components in the θ-axis, i.e. $\tau_{r\theta}(r, z)$ and $\tau_{\theta z}(r, z)$ existing in the joint (Timoshenko and Goodier, 1951) and the two stress components are related by the equilibrium equation

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 \tau_{r\theta}) + \frac{\partial}{\partial \zeta}\tau_{\theta z} = 0 \qquad (1)$$

in which non-dimensional coordinates $\rho = r/R$, $\zeta = z/R$ are introduced for simplifying the forthcoming operations and conclusions. We note that when any one of the two stress components in eqn (1) is known, the other can be easily derived with proper integration procedure.

Let $(\tau_{r\theta 1}, \tau_{\theta z 1})$, $(\tau_{r\theta 2}, \tau_{\theta z 2})$ and $(\tau_{r\theta 3}, \tau_{\theta z 3})$ be the stress components in the inner adherend, the outer adherend and the adhesive layer of the jointed portion, respectively. As the basis of the analysis to be developed, a general pattern of stress distribution which satisfies eqn (1), conditions for stress continuity across the bonding surfaces $\rho = \rho_1 = R_1/R$, $\rho = \rho_2 = R_2/R$ as well as those boundary stress conditions of the joint are constructed in this section. It starts with two assumptions:

(1) The stress components $\tau_{\theta z 1}$, $\tau_{\theta z 2}$ in the inner and outer adherends of the joint may be considered varying linearly with the coordinate ρ (Fig. 2), i.e.

$$\tau_{\theta z 1} = \rho \tau_1(\zeta) \quad \left(\rho_0 \leqslant \rho \leqslant \rho_1, \rho_0 = \frac{R_0}{R}\right) \qquad (2)$$

$$\tau_{\theta z 2} = \rho \tau_2(\zeta) \quad (\rho_2 \leqslant \rho \leqslant 1) \qquad (3)$$

where $\tau_1(\zeta)$ and $\tau_2(\zeta)$ are two unknown functions yet to be determined, and there

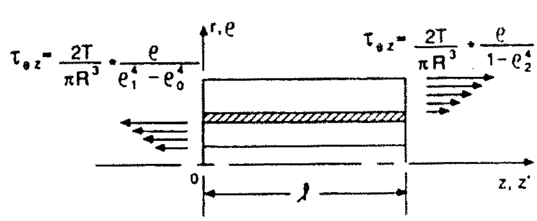

Fig. 2. Distribution of shearing stress at $z = 0$ and $z = l$.

will be two boundary conditions (at $\zeta = 0$ and $\zeta = \lambda = l/R$) imposed on each of them

$$\tau_1(0) = \frac{2T}{\pi R^3}\cdot\frac{1}{\rho_1^4-\rho_0^4}, \quad \tau_1(\lambda) = 0 \tag{4a}$$

$$\tau_2(0) = 0, \qquad \tau_2(\lambda) = \frac{2T}{\pi R^3}\cdot\frac{1}{1-\rho_2^4}. \tag{4b}$$

(2) The stress component $\tau_{\theta z 3}$ in the adhesive layer may be taken as vanishing, i.e.

$$\tau_{\theta z 3} = 0 \quad (\rho_1 \leqslant \rho \leqslant \rho_2) \tag{5}$$

(and the free end conditions at $\zeta = 0$, $\zeta = \lambda$ for the adhesive layer are therefore automatically satisfied).

These two assumption are quite similar to those adopted by Volkersen–Adams' analysis (1965f, 1977c), but the crucial feature of the present analysis lies in the further deduction and consideration for the stress components $\tau_{r\theta 1}$, $\tau_{r\theta 2}$, which are neglected in Volkersen–Adams' analysis (1965g, 1977d).

From eqns (1), (2), (3) and also the traction-free conditions on the inner ($\rho = \rho_0$) and outer ($\rho = 1$) surfaces of the joint, it is readily derived that in the inner and outer adherends, the stress components $\tau_{r\theta 1}$ and $\tau_{r\theta 2}$ must be

$$\tau_{r\theta 1} = \left(\frac{\rho_0^4-\rho^4}{4\rho^2}\right)\frac{d\tau_1}{d\zeta} \quad (\rho_0 \leqslant \rho \leqslant \rho_1) \tag{6}$$

$$\tau_{r\theta 2} = \left(\frac{1-\rho^4}{4\rho^2}\right)\frac{d\tau_2}{d\zeta} \quad (\rho_2 \leqslant \rho \leqslant 1) \tag{7}$$

respectively. Applying eqns (1) and (5) and taking into account the condition for stress continuity across the bonding surface $\rho = \rho_1$ (i.e. $\tau_{r\theta 1}|_{\rho_1} = \tau_{r\theta 3}|_{\rho_1}$), the stress component $\tau_{r\theta 3}$ in the adhesive layer can also be deduced:

$$\tau_{r\theta 3} = \left(\frac{\rho_0^4-\rho_1^4}{4\rho^2}\right)\frac{d\tau_1}{d\zeta} \quad (\rho_1 \leqslant \rho \leqslant \rho_2). \tag{8}$$

As for the condition of continuity of stress $\tau_{r\theta 2}/\rho_2 = \tau_{r\theta 3}/\rho_2$ across the bonding surface $\rho = \rho_2$, we obtain from eqns (7) and (8)

$$\left(\frac{1-\rho_2^4}{4\rho_2^2}\right)\frac{d\tau_2}{d\zeta} = \left(\frac{\rho_0^4-\rho_1^4}{4\rho_2^2}\right)\frac{d\tau_1}{d\zeta}$$

which can be seen as a condition of constraint imposed on τ_1 and τ_2. Through integration and referring to the end conditions (4a), (4b), this constraint is equivalent to

$$\alpha_1\tau_1(\zeta)+\alpha_2\tau_2(\zeta) = \tau_0 \tag{9}$$

where

$$\alpha_1 = \rho_1^4 - \rho_0^4, \quad \alpha_2 = 1 - \rho_2^4, \quad \tau_0 = \frac{2T}{\pi R^3}. \tag{10}$$

Thus, the stress distribution in the joint block may now be determined by only one of the two unknown functions τ_1, τ_2.

GOVERNING EQUATION FOR STRESS DISTRIBUTION

To obtain further equations for the determination of τ_1 (or τ_2), use has to be made of the compatibility of deformation of the joint, and this can be done in a rational manner by means of the variational theorem of complementary energy (Washizu, 1968). There are three distinct parts of the joint and the total complementary energy of the joint to be minimized may be written as

$$U(\tau_{r\theta}, \tau_{\theta z}) = U_1 + U_2 + U_3 \tag{11}$$

and

$$U_1 = \frac{R^3}{2G_1} \int_0^{\lambda} \int_{\rho_0}^{\rho_1} (\tau_{r\theta 1}^2 + \tau_{\theta z 1}^2) 2\pi\rho \, d\rho \, d\zeta \tag{11a}$$

$$U_2 = \frac{R^3}{2G_2} \int_0^{\lambda} \int_{\rho_2}^{1} (\tau_{r\theta 2}^2 + \tau_{\theta z 2}^2) 2\pi\rho \, d\rho \, d\zeta \tag{11b}$$

$$U_3 = \frac{R^3}{2G_3} \int_0^{\lambda} \int_{\rho_1}^{\rho_2} (\tau_{r\theta 3}^2 + \tau_{\theta z 3}^2) 2\pi\rho \, d\rho \, d\zeta. \tag{11c}$$

After substituting expressions (2), (3), (5), (6), (7) and (8) for $\tau_{r\theta 1}, \ldots, \tau_{\theta z 3}$ into (11) and carrying out the integration with respect to ρ, it is found that

$$U_1 = \frac{\pi R^3}{G_3} \int_0^{\lambda} \left[A_1 \left(\frac{d\tau_1}{d\zeta} \right)^2 + B_1 \tau_1^2 \right] d\zeta, \quad U_2 = \frac{\pi R^3}{2G_3} \int_0^{\lambda} \left[A_2 \left(\frac{d\tau_2}{d\zeta} \right)^2 + B_2 \tau_2^2 \right] d\zeta,$$

$$U_3 = \frac{\pi R^3}{G_3} \int_0^{\lambda} A_3 \left(\frac{d\tau_1}{d\zeta} \right)^2 d\zeta$$

and

$$U = \frac{\pi R^3}{G_3} \int_0^{\lambda} \left[(A_1 + A_3) \left(\frac{d\tau_1}{d\zeta} \right)^2 + A_2 \left(\frac{d\tau_2}{d\zeta} \right)^2 + B_1 \tau_1^2 + B_2 \tau_2^2 \right] d\zeta \tag{12}$$

in which

$$A_1 = \frac{G_3}{G_1} \left[\frac{\rho_0^8}{32} (\rho_0^{-2} - \rho_1^{-2}) + \frac{\rho_0^4}{16} (\rho_0^2 - \rho_1^2) + \frac{1}{96} (\rho_1^6 - \rho_0^6) \right],$$

$$B_1 = \frac{1}{4} \left(\frac{G_3}{G_1} \right) (\rho_1^4 - \rho_0^4),$$

$$A_2 = \frac{G_3}{G_2} \left[\frac{1}{32} (\rho_2^{-2} - 1) + \frac{1}{16} (\rho_2^2 - 1) + \frac{1}{96} (1 - \rho_2^6) \right],$$

$$B_2 = \frac{1}{4} \left(\frac{G_3}{G_2} \right) (1 - \rho_2^4),$$

$$A_3 = \frac{1}{32} (\rho_0^4 - \rho_1^4)^2 (\rho_1^{-2} - \rho_2^{-2}) \tag{13}$$

are all non-dimensional coefficients. We note that in the integrals U_1, U_2 and U_3, the terms with coefficients A_1, A_2 and A_3 represent the complementary energy due to $\tau_{r\theta 1}$, $\tau_{r\theta 2}$ and $\tau_{r\theta 3}$, respectively (6, 7, 8).

By using the constraint (9) to eliminate τ_2 in the integrand of (12), we obtain

$$U = \frac{\pi R^3}{G_3}\int_0^{\lambda}\left\{\left[A_1+A_3+A_2\left(\frac{\alpha_1}{\alpha_2}\right)^2\right]\left(\frac{d\tau_1}{d\zeta}\right)^2+\left[B_1+B_2\left(\frac{\alpha_1}{\alpha_2}\right)^2\right]\tau_1^2 - 2B_2\left(\frac{\alpha_1\tau_0}{\alpha_2^2}\right)\tau_1+B_2\left(\frac{\tau_0}{\alpha_2}\right)^2\right\}d\zeta. \quad (14)$$

Then, carrying out the variation $\delta U = 0$ under the given two end conditions (4a), the governing equation for τ_1 which renders the total complementary energy U a minimum is established:

$$\left[A_1+\left(\frac{\alpha_1}{\alpha_2}\right)^2 A_2+A_3\right]\frac{d^2\tau_1}{d\zeta^2}-\left[B_1+\left(\frac{\alpha_1}{\alpha_2}\right)^2 B_2\right]\tau_1 = -B_2\left(\frac{\alpha_1}{\alpha_2^2}\right)\tau_0. \quad (15)$$

SOLUTION

The general solution of eqn (15) may be written as

$$\tau_1(\zeta) = C_1\, e^{-k\zeta}+C_2\, e^{-k(\lambda-\zeta)}+\frac{\left(\frac{\alpha_1}{\alpha_2}\right)^2 B_2}{B_1+\left(\frac{\alpha_1}{\alpha_2}\right)^2 B_2}\cdot\frac{\tau_0}{\alpha_1} \quad (16)$$

in which

$$k = \sqrt{\frac{B_1+\left(\frac{\alpha_1}{\alpha_2}\right)^2 B_2}{A_1+\left(\frac{\alpha_1}{\alpha_2}\right)^2 A_2+A_3}} \quad (17)$$

and C_1, C_2 are two constants of integration which can be readily determined by two end conditions (4a).

Calculation shows that for most cases of practical importance, $k\lambda$ is generally much greater than 1 (unless the joint is unusually "short" $l \ll R$), so it is preferable to split the general solution (16) into two parts, i.e.

(A) for the left end zone of the joint (near $\zeta = 0$)

$$\tau_1(\zeta) = C_1\, e^{-k\zeta}+\frac{\left(\frac{\alpha_1}{\alpha_2}\right)^2 B_2}{B_1+\left(\frac{\alpha_1}{\alpha_2}\right)^2 B_2}\cdot\frac{\tau_0}{\alpha_1}$$

$$= \frac{B_1}{B_1+\left(\frac{\alpha_1}{\alpha_2}\right)^2 B_2}\frac{\tau_0}{\alpha_1}e^{-k\zeta}+\frac{\left(\frac{\alpha_1}{\alpha_2}\right)^2 B_2}{B_1+\left(\frac{\alpha_1}{\alpha_2}\right)^2 B_2}\frac{\tau_0}{\alpha_1}. \quad (18)$$

(B) for the right end zone of the joint (near $\zeta = \lambda$)

$$\tau_1(\zeta) = C_2\,\mathrm{e}^{-k(\lambda-\zeta)} + \frac{\left(\dfrac{\alpha_1}{\alpha_2}\right)^2 B_2}{B_1 + \left(\dfrac{\alpha_1}{\alpha_2}\right)^2 B_2} \cdot \frac{\tau_0}{\alpha_1}$$

$$= \frac{\left(\dfrac{\alpha_1}{\alpha_2}\right)^2 B_2}{B_1 + \left(\dfrac{\alpha_1}{\alpha_2}\right)^2 B_2} \cdot \frac{\tau_0}{\alpha_1}(1 - \mathrm{e}^{-k(\lambda-\zeta)}). \tag{19}$$

We note that in solutions (18) and (19), the two end conditions (4a) have been taken into account.

From eqns (8), (18) and (19), the shearing stresses in the two end zones of the adhesive layer are finally obtained:

$$\tau_{r\theta 3} = \begin{cases} k \cdot \dfrac{B_1}{B_1 + \left(\dfrac{\alpha_1}{\alpha_2}\right)^2 B_2} \cdot \dfrac{\tau_0}{4\rho^2}\,\mathrm{e}^{-k\zeta} & (\text{near } \zeta = 0) \\[2ex] k \cdot \dfrac{\left(\dfrac{\alpha_1}{\alpha_2}\right)^2 B_1}{B_1 + \left(\dfrac{\alpha_1}{\alpha_2}\right)^2 B_2} \cdot \dfrac{\tau_0}{4\rho^2}\,\mathrm{e}^{-k(\lambda-\zeta)} & (\text{near } \zeta = \lambda). \end{cases} \tag{20}$$

DISCUSSION

We are mainly interested in $\tau_{r\theta 3}$ (20). From (20) and the previous analysis as well, it is easily seen that:

(1) The maximum shearing stress in the adhesive layer always occurs at the two ends of the joint ($\zeta = 0$ and $\zeta = \lambda$), and acts on the inner bonding surface ($\rho = \rho_1$) of the layer. More specifically, when $B_1 > (\alpha_1/\alpha_2)^2 B_2$ happens, it is the left end ($\zeta = 0$) where the maximum shearing stress appears and this maximum value is

$$\tau_{r\theta 3}|_{\max} = \frac{k \cdot B_1}{B_1 + \left(\dfrac{\alpha_1}{\alpha_2}\right)^2 B_2} \cdot \frac{\tau_0}{4\rho_1^2} \tag{21}$$

and when $B_1 < (\alpha_1/\alpha_2)^2 B_2$, the maximum shearing stress occurs at the right end ($\zeta = \lambda$) and this maximum value is

$$\tau_{r\theta 3}|_{\max} = \frac{k \cdot \left(\dfrac{\alpha_1}{\alpha_2}\right)^2 B_2}{B_1 + \left(\dfrac{\alpha_1}{\alpha_2}\right)^2 B_2} \cdot \frac{\tau_0}{4\rho_1^2}. \tag{22}$$

(2) Expression (20) shows that the decaying exponent k (17) is precisely the index for

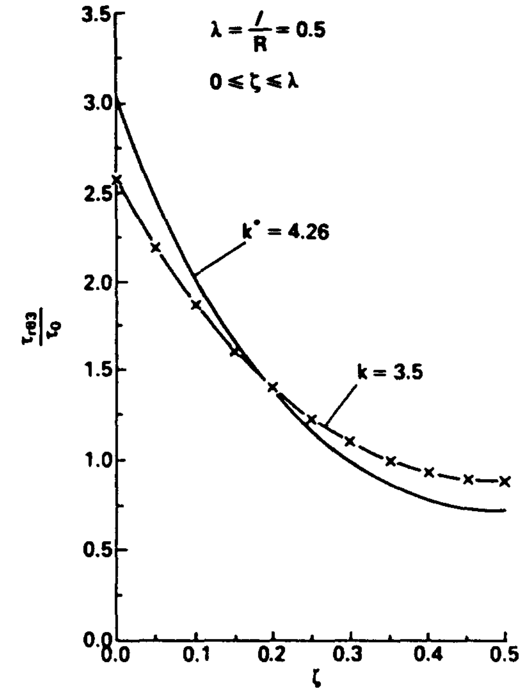

Fig. 3. Distribution of $\tau_{r\theta 3}$ for $l = 0.5R$.

the influence which the adhesive layer may have on the degree of the end stress concentration when the dimensions and elastic coefficients of the adherends of a joint are given.

(3) Examining the expression for k (17) it is seen that with the increase of the adhesive modulus G_3 (13) and/or the decreasing of the thickness of the adhesive layer the value of k

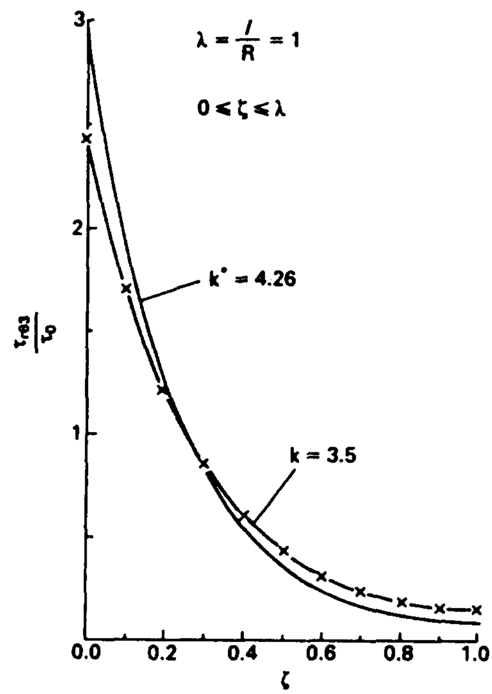

Fig. 4. Distribution of $\tau_{r\theta 3}$ for $l = R$.

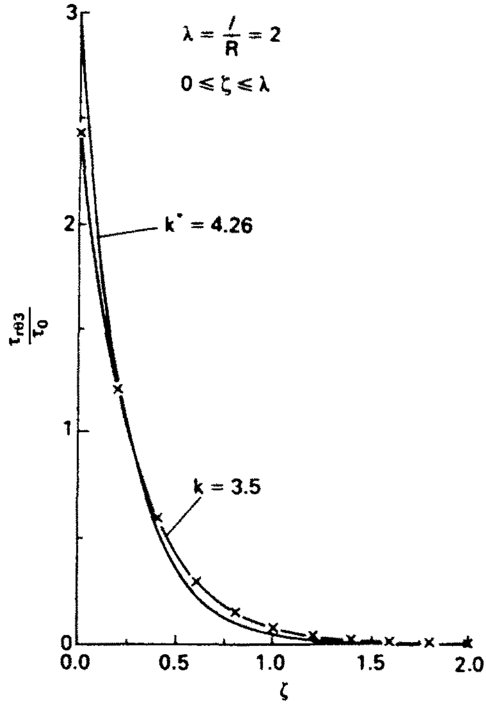

Fig. 5. Distribution of $\tau_{r\theta 3}$ for $l = 2R$.

increases (17, 13) and hence the stress concentration at the two ends of the adhesive layer (21, 22) also increases with the increasing value of k.

(4) In Volkersen–Adams' analysis (1965h, 1977e), the stress components $\tau_{r\theta 1}$, $\tau_{r\theta 2}$ existing in the two adherends are disregarded. This is equivalent to neglecting the contribution of $\tau_{r\theta 1}$ and $\tau_{r\theta 2}$ to the complementary energy expressions (11a) and (11b), and hence results in the change of the first coefficient in the governing eqn (15) from $[A_1+(\alpha_1/\alpha_2)^2A_2+A_3]$ to A_3. Obviously, this negligence can only be justified with the precondition

$$\left[A_1+\left(\frac{\alpha_1}{\alpha_2}\right)^2 A_2\right] \ll A_3 \tag{23}$$

and referring to the definitions for A_1, A_2 and A_3 (13), it is seen that this pre-condition actually means that the adhesive layer of the joint must be rather "soft" ($G_3 \ll G_1, G_2$) when compared with the two adherends. Thus, the validity of Volkersen–Adams' analysis (1965i, 1977f) is limited to the case of the "soft" adhesive layer.

According to Volkersen–Adams' analysis (1965j, 1977g), the above mentioned index of the stress concentration in the end zones of the adhesive layer becomes

$$k^* = \sqrt{\frac{B_1+\left(\frac{\alpha_1}{\alpha_2}\right)^2 B_2}{A_3}}. \tag{24}$$

Undoubtedly, the index may always give a higher estimation of the stress concentration than what is predicted by the present analysis, and for a joint with a "stiff" adhesive layer, the difference between k and k^* may be significant.

For example, taking $\rho_0 = 0$, $\rho_1 = 0.60$, $\rho_2 = 0.62$, $G_3/G_1 = G_3/G_2 = 0.045$ (epoxy/aluminum) we have

$$A_1 = 2.19\times10^{-5},\quad A_2 = 9.59\times10^{-4},\quad A_3 = 9.26\times10^{-5},$$
$$B_1 = 1.46\times10^{-3},\quad B_2 = 9.59\times10^{-3},$$
$$\alpha_1 = 0.13,\qquad \alpha_2 = 0.85,$$

hence $A_1+(\alpha_1/\alpha_2)^2A_2 = 4.43\times10^{-5}$ and $k = 3.50$, $k^* = 4.26$.

The value of k^* is twenty per cent higher than that of k. This example illustrates the earlier statement that when the condition (23) is not satisfied as in this example, the difference between the two indices k and k^* can be significant. Using the preceding data, from eqns (8), (16) and (4a) we obtain

$$\frac{\tau_{r\theta 3}}{\tau_0} = 0.09k[C_1\,e^{-k\zeta}-C_2\,e^{-k(\lambda-\zeta)}] \tag{25}$$

in which

$$C_1 = (1-e^{-2k\lambda})^{-1}[7.716-7.643\times10^{-3}(1-e^{-k\lambda})]$$
$$C_2 = (1-e^{-2k\lambda})^{-1}[-7.716\,e^{-k\lambda}-7.643\times10^{-3}(1-e^{-k\lambda})].$$

The distributions of $\tau_{r\theta 3}$ from $\zeta = 0$ to $\zeta = \lambda$ for $\lambda = 0.5$, 1 and 2 are shown in Figs 3, 4 and 5. In each figure, two curves for $k = 3.5$ and $k^* = 4.26$ are plotted. These figures show that the values of $\tau_{r\theta 3}/\tau_0$ for $k^* = 4.26$ are 18–20% higher than that for $k = 3.5$. Hence the difference between the two sets of solutions of $\tau_{r\theta 3}/\tau_0$ for $k = 3.5$ and $k^* = 4.26$ can be significant.

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